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ON THE NABLA OF QUATERNIONS.

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In studying McAulay's differential linear vector function, I came across the nablas whose arguments are the sums of vectors, which led me to consider the same operator in its wider sense, thereby extending it to the case of quaternion argument with transformations that might be useful in attacking the problems of physics and geometry. The discussion is not exhaustive, but it is hoped that the prominent features of the operator have been touched upon. Along with the quaternion treatment are appended the results in semi-Cartesian forms, thereby showing how much simplicity is gained by the former treatment.

Throughout the paper, ρ, σ, q, p have the following meanings whenever they are used in semi-Cartesian treatment:—

$$\rho = ix + jy + kz\,,$$

$$\sigma = iX + jY + kZ\,,$$

$$q = t + ix + jy + kz\,,$$

$$p = T + iX + jY + kZ\,.$$

Occasionally the same symbols are used with different meanings in different divisions of the paper; these will, however, be carefully defined or explained in each case.

§ 1.

In stating the differential symbols in the original definition of a nabla, two kinds of notations are used; thus, for example,

$$\left.\begin{aligned} {}_\rho\nabla &= i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\,, \\ {}_\rho\nabla &= i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\,, \end{aligned}\right\} \qquad (1)$$

These are immaterial when ρ is the independent variable vector, i. e. when x, y, z are the three independent scalar variables; but when we come to generalize the same operator we are restricted to one of these two notations.

Thus, for reasons to be seen later on, I propose to write down for any vectors ρ and σ,

$$\left.\begin{aligned} {}_\rho\nabla &= i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}, \\ {}_\sigma\nabla &= i\frac{\partial}{\partial X} + j\frac{\partial}{\partial Y} + k\frac{\partial}{\partial Z}; \end{aligned}\right\} \quad \text{(A)}$$

where $\dfrac{\partial Q}{\partial x}$, etc., represent *partial* differential coefficients of a certain function Q.

Quite reasonably we may extend the nabla in the following form :—

$$\left.\begin{aligned} {}_q\nabla &= \frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}, \\ {}_{Kq}\nabla &= \frac{\partial}{\partial t} - i\frac{\partial}{\partial x} - j\frac{\partial}{\partial y} - k\frac{\partial}{\partial z}. \end{aligned}\right\} \quad \text{(B)}$$

When q is a function of t, or when it is an independent variable quaternion, that is, when t, x, y, z are the four independent *variable scalars*, then invariably we have

$$d = SdKq\,{}_q\nabla = Sdq\,{}_{Kq}\nabla\,; \tag{2}$$

and this is the form usually adopted for the case when q degenerates into a vector.

If, on the other hand, we were to express ${}_\sigma\nabla$, ${}_q\nabla$ in the form of total differential coefficients,

$$ {}_q\nabla = \frac{d}{dt} + i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}, $$

we could obtain the form (2) only when t, x, y, z are independent ; if q were a function of t, we should have

$$ SdKq\,{}_q\nabla = dt\left[\frac{d}{dt} + \left[\frac{dx}{dt}\right]^2\frac{\partial}{\partial x} + \left[\frac{dy}{dt}\right]^2\frac{\partial}{\partial y} + \left[\frac{dz}{dt}\right]^2\frac{\partial}{\partial z}\right]; $$

while by definitions (A) and (B), even when q is a function of t,

$$\begin{aligned} d &= SdKq\,{}_q\nabla \\ &= dt\left[\frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}\right]. \end{aligned} \tag{3}$$

As illustrations of the advantage of using definitions (A) and (B) may be

cited the operator employed in hydrodynamics, which is obtained in the form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial y} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}$$

$$= S\frac{dKq}{dt}\,{}_q\nabla = S\frac{dq}{dt}\,{}_{Kq}\nabla\,; \tag{4}*$$

and Euler's equations, which are written at once in the form,

$$(Vdq)_1\; S\frac{dKq}{dt}\,{}_q\nabla_1 = {}_{Vq}\nabla\,\Omega dt\,. \tag{5}*$$

Thus it is seen that nablas in their extended form have direct physical applications, not to mention the "electrodynamic quaternion" of Hamilton.

When q degenerates into a scalar t, to be consistent we must write

$$_t\nabla = \frac{\partial}{\partial t}, \tag{6}$$

and when t is the only independent variable, it is immaterial whether this be written in the form of a partial or a total differential symbol.

§ 2.

Usually the differential operator is written in the following forms, when q, ρ, σ, and t occurring therein are each taken for *the sole independent variable;* and these are taken as new definitions of nablas :—

$$\begin{aligned} d &= SdKq\,{}_q\nabla = Sdq\,{}_{Kq}\nabla \\ &= SdK\rho\,{}_\rho\nabla = Sd\rho\,{}_{K\rho}\nabla \\ &= SdK\sigma\,{}_\sigma\nabla = Sd\sigma\,{}_{K\sigma}\nabla \\ &= SdKt\,{}_t\nabla = Sdt\,{}_{Kt}\nabla\,, \end{aligned} \tag{7}$$

where it is to be noticed that when q degenerates into a vector or a scalar,

$$\left.\begin{aligned} K\rho &= -\rho\,, & {}_{K\rho}\nabla &= -{}_\rho\nabla = K{}_\rho\nabla\,; \\ K\sigma &= -\sigma\,, & {}_{K\sigma}\nabla &= -{}_\sigma\nabla = K{}_\sigma\nabla\,; \\ Kt &= t\,, & {}_{Kt}\nabla &= {}_t\nabla\,; \end{aligned}\right\} \tag{8}$$

and, in general,

$$_{Kq}\nabla = K{}_q\nabla\,.$$

* Kirchhoff, Mechanik, S. 164. The subscripts, 1, show that Vdq is the quantity to which ${}_q\triangle$ applies.

Similarly with respect to Laplace's operator, we usually call ∇^2 the negative of the operator, but since

$$\left.\begin{aligned} S_q\nabla_{Kq}\nabla = S_{Kq}\nabla_q\nabla &= \frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \\ \text{we have, as may readily be seen,}& \\ S_q\nabla_q\nabla = S_{Kq}\nabla_{Kq}\nabla &= \frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2} \end{aligned}\right\}, \tag{9}$$

In reality, the positive of the Laplace's operator is

$$S_\rho\nabla_{K\rho}\nabla = S_{K\rho}\nabla_\rho\nabla\,. \tag{10}$$

§ 3.

For the change of the independent vector or quaternion variables, if we take σ to be a vector function of ρ, and p to be a quaternion function of q, we have always

$$\left.\begin{aligned} d\sigma &= \varphi d\rho\,, & dp &= \Phi dq\,, \\ d\rho &= \varphi^{-1}d\sigma\,; & dq &= \Phi^{-1}dp\,, \end{aligned}\right\} \tag{11}$$

where φ, Φ are respectively a linear vector and a linear quaternion function.

Then, since

$$\begin{aligned} d = Sd\rho_{K\rho}\nabla &= S\varphi^{-1}d\sigma_{K\rho}\nabla = Sd\sigma\varphi'^{-1}{}_{K\rho}\nabla \\ &= Sd\sigma_{K\sigma}\nabla = S\varphi d\rho_{K\sigma}\Delta \ = Sd\rho\varphi'_{K\sigma}\nabla\,, \end{aligned}$$

and

$$\begin{aligned} d = Sdq_{Kq}\nabla &= S\Phi^{-1}dp_{Kq}\nabla = Sdp\Phi'^{-1}{}_{Kq}\nabla \\ &= Sdp_{Kp}\nabla = S\Phi dq_{Kp}\nabla \ = Sdq\Phi'_{Kp}\nabla\,, \end{aligned}$$

we obtain the relations

$$\left.\begin{aligned} {}_\rho\nabla &= \varphi'{}_\sigma\nabla\,, \\ {}_\sigma\nabla &= \varphi'^{-1}{}_\rho\nabla\,, \end{aligned}\right\} \tag{12*}$$

$$\left.\begin{aligned} {}_{Kq}\nabla = \Phi'{}_{Kp}\nabla = \Phi' K_p\nabla \quad &\text{or} \quad {}_q\nabla = \Phi'{}_p\nabla\,, \\ {}_{Kp}\nabla = \Phi'^{-1}{}_{Kq}\nabla = \Phi'^{-1}K_q\nabla \quad &\text{or} \quad {}_p\nabla = \Phi'^{-1}{}_q\nabla\,. \end{aligned}\right\} \tag{13}$$

It is almost needless to remark that the nablas of quaternion, vector, and

* Tait, Quaternions, 3rd ed., p. 373; φ', Φ' being conjugates of φ, Φ.

scalar arguments are respectively quaternion, vector, and scalar operators, of which the first is the sum of the last two :—

$$\left.\begin{aligned} {}_q\nabla &= S_q\nabla + V_q\nabla\,, \\ S_q\nabla &= S_{Kq}\nabla = {}_t\nabla\,, \\ V_q\nabla &= -V_{Kq}\nabla = -VK_q\nabla = {}_{Vq}\nabla = {}_\rho\nabla\,. \end{aligned}\right\} \tag{14}$$

The transformation of Laplace's operator is effected by

$$\left.\begin{aligned} S_\rho\nabla_{K\rho}\nabla &= S_\sigma\nabla\varphi\varphi'_{K\sigma}\nabla\,, \\ S_q\nabla_{Kq}\nabla &= S_p\nabla\Phi\Phi'_{Kp}\nabla = S_{Kp}\nabla\Phi\Phi'_p\nabla, \\ \text{and}\qquad S_q\nabla_q\nabla &= S(\Phi'_p\nabla)^2 = S_p\nabla\Phi\Phi'_p\nabla\,, \end{aligned}\right\} \tag{15}$$

where $\varphi\varphi'$ and $\Phi\Phi'$ are self conjugate functions.

Thus, for example, when

$$\sigma = W\rho\,,$$

where W is a scalar function of ρ, then

$$d\sigma = -W_1\rho S\nabla_1 d\rho + Wd\rho\,,$$

in which W_1 is not a factor, but the quantity to which the operation ∇_1 applies, which is shown by the common suffix.* Hence

$$\begin{aligned} \varphi\lambda &= W\lambda - W_1\rho S\nabla_1\lambda\,, \\ \varphi'\lambda &= W\lambda - \nabla WS\rho\lambda\,, \\ \psi\lambda &= W^2\lambda - WV(\nabla WV\rho\lambda)\,, \\ S\varphi'\lambda\psi\mu : S\lambda\mu &= W^2(W - S\rho\nabla W)\,, \end{aligned}$$

where λ is any quaternion, scalar, or vector, ψ is Hamilton's inverse function, and ∇ is used for ${}_\rho\nabla$ when it is already operated, and we have

$$\left.\begin{aligned} {}_{W\rho}\nabla &= \varphi'^{-1}{}_\rho\nabla = (W_\rho\nabla + \nabla WS\rho_\rho\nabla - {}_\rho\nabla S\rho\nabla W) : W(W - S\rho\nabla W), \\ {}_\rho\nabla &= \varphi'{}_{W\rho}\nabla = W_{W\rho}\nabla - \nabla WS\rho_{W\rho}\nabla\,, \\ S_\rho\nabla_{K\rho}\nabla &= -W^2S_{W\rho}\nabla_{W\rho}\nabla + 2WS\nabla W_{W\rho}\nabla S\rho_{W\rho}\nabla - (\nabla W)^2S^2\rho_{W\rho}\nabla\,. \end{aligned}\right\} \tag{16}$$

The case $\sigma = \rho^n$ $(n = 2m + 1)$ is included in the above, for here we have

$$W = (-1)^m(T\rho)^{2m}\,,$$

* See McAulay, Utility of Quaternions in Physics, p. 13.

where W is a constant, equal to n; whence $\nabla W = 0$, and

$$ {}_{n\rho}\nabla = \frac{1}{n}\,{}_{\rho}\nabla \,. $$

§ 4.

The results of the above transformations may be expressed in a different form without the use of φ; for we may write, in general,

$$ d\sigma = -\tfrac{1}{3}\left(\sigma_1 S\nabla_1 d\rho + \sigma_2 S\nabla_2 d\rho + \sigma_3 S\nabla_3 d\rho\right). $$

Hence φ of the last section is of the form,

$$ \varphi\tau = -\tfrac{1}{3}\left(\sigma_1 S\nabla_1\tau + \sigma_2 S\nabla_2\tau + \sigma_3 S\nabla_3\tau\right). \tag{17} $$

Whence, the third degree invariant m and the conjugate of Hamilton's inverse function of φ are

$$ m = \tfrac{1}{6}\, S\sigma_1\sigma_2\sigma_3\, S\nabla_1\nabla_2\nabla_3 \,, $$

$$ \begin{aligned} \psi'\tau &= \tfrac{1}{6}\left(V\sigma_1\sigma_2\, S\nabla_2\nabla_1\tau + V\sigma_2\sigma_3\, S\nabla_3\nabla_2\tau + V\sigma_3\sigma_1\, S\nabla_1\nabla_3\tau\right) \\ &= \tfrac{1}{2}\, V\sigma_1\sigma_2\, S\nabla_2\nabla_1\tau \,; \end{aligned} $$

also

$$ \begin{aligned} \varphi'\tau &= -\tfrac{1}{3}\left(\nabla_1 S\sigma_1\tau + \nabla_2 S\sigma_2\tau + \nabla_3 S\sigma_3\tau\right) \\ &= -\nabla_1 S\sigma_1\tau \,. \end{aligned} $$

Hence we obtain the following results of transformation :—

$$ \left.\begin{aligned} {}_{\rho}\nabla &= \varphi'{}_{\sigma}\nabla = -\nabla_1\, S\sigma_{1\sigma}\nabla \,, \\ {}_{\sigma}\nabla &= \varphi'^{-1}{}_{\sigma}\nabla = m^{-1}\psi'{}_{\rho}\nabla \\ &= 3\left[\, V\sigma_1\sigma_2\, S_{\rho}\nabla\nabla_1\nabla_2\right] : \left[S\sigma_1\sigma_2\sigma_3\, S\nabla_1\nabla_2\nabla_3\right] , \\ S_{\rho}\nabla K_{\rho}\nabla &= -S\nabla_1\nabla_2\, S\sigma_{1\sigma}\nabla\, S\sigma_{2\sigma}\nabla \,. \end{aligned}\right\} \tag{18} $$

In these expressions the suffixes applied to σ and ∇ are due to McAulay and mean that the differential operator ∇ applies to σ of the same suffix while as a vector ∇ remains in its place. ∇ is used for ${}_{\rho}\nabla$ when the same is already applied to vectors σ in the expressions, while ${}_{\rho}\nabla$ remains in its nascent state. The brackets in the expression for ${}_{\sigma}\nabla$ indicate that the dividend and divisor are independent with respect to the suffixes, 1, 2, occurring in both.

The semi-Cartesian forms for the above expressions are as follows :—

$$\left.\begin{aligned}
S\sigma_1\sigma_2\sigma_3\, S\nabla_1\nabla_2\nabla_3 &= -\,6m = -\,6S\frac{\partial\sigma}{\partial x}\frac{\partial\sigma}{\partial y}\frac{\partial\sigma}{\partial z}\,,\\
V\sigma_1\sigma_2\, S\nabla_2\nabla_1\tau &= -\,3\,\psi'\tau\\
&= -\,2\left[V\frac{\partial\sigma}{\partial x}\frac{\partial\sigma}{\partial y}Sk\tau + V\frac{\partial\sigma}{\partial y}\frac{\partial\sigma}{\partial z}Si\tau + V\frac{\partial\sigma}{\partial z}\frac{\partial\sigma}{\partial x}Sj\tau\right],\\
\sigma_1 S\nabla_1\tau &= -\,3\varphi\tau = \frac{\partial\sigma}{\partial x}Si\tau + \frac{\partial\sigma}{\partial y}Sj\tau + \frac{\partial\sigma}{\partial z}Sk\tau\,,\\
\nabla_1 S\sigma_1\tau &= -\,3\varphi'\tau = iS\tau\frac{\partial\sigma}{\partial x} + jS\tau\frac{\partial\sigma}{\partial y} + kS\tau\frac{\partial\sigma}{\partial z}\,,
\end{aligned}\right\} \quad (19)$$

and so on. Transformations of quarternion nablas are treated in the same way.

§ 5.

If we try to deduce the results of the last paragraph from purely Cartesian forms, we arrive at somewhat novel but complicated expressions. Thus, supposing σ to be a function of ρ, and consequently vice versa, we have as usual

$$\left.\begin{aligned}
dX &= \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy + \frac{\partial X}{\partial z}dz\,,\\
dY &= \frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy + \frac{\partial Y}{\partial z}dz\,,\\
dZ &= \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy + \frac{\partial Z}{\partial z}dz\,.
\end{aligned}\right\} \quad (20)$$

Solving this system of equations we obtain

$$\left.\begin{aligned}
M_\sigma dx &= M_{Xx}dX + M_{Yx}dY + M_{Zx}dZ\,,\\
M_\sigma dy &= M_{Xy}dX + M_{Yy}dY + M_{Zy}dZ\,,\\
M_\sigma dz &= M_{Xz}dX + M_{Yz}dY + M_{Zz}dZ\,,
\end{aligned}\right\} \quad (21)$$

where M_σ is the determinant

$$M_\sigma = \begin{vmatrix}
\dfrac{\partial X}{\partial x}, & \dfrac{\partial X}{\partial y}, & \dfrac{\partial X}{\partial z}\\[2ex]
\dfrac{\partial Y}{\partial x}, & \dfrac{\partial Y}{\partial y}, & \dfrac{\partial Y}{\partial z}\\[2ex]
\dfrac{\partial Z}{\partial x}, & \dfrac{\partial Z}{\partial y}, & \dfrac{\partial Z}{\partial z}
\end{vmatrix}, \quad (22)$$

and M_{Xx}, etc., are its minors. Consequently

$$\left.\begin{aligned}
\frac{\partial x}{\partial X} &= \frac{M_{Xx}}{M_\sigma}, \quad \frac{\partial x}{\partial Y} = \frac{M_{Yx}}{M_\sigma}, \quad \frac{\partial x}{\partial Z} = \frac{M_{Zx}}{M_\sigma}, \\
\frac{\partial y}{\partial X} &= \frac{M_{Xy}}{M_\sigma}, \quad \frac{\partial y}{\partial Y} = \frac{M_{Yy}}{M_\sigma}, \quad \frac{\partial y}{\partial Z} = \frac{M_{Zy}}{M_\sigma}, \\
\frac{\partial z}{\partial X} &= \frac{M_{Xz}}{M_\sigma}, \quad \frac{\partial z}{\partial Y} = \frac{M_{Yz}}{M_\sigma}, \quad \frac{\partial z}{\partial Z} = \frac{M_{Zz}}{M_\sigma}.
\end{aligned}\right\} \tag{23}$$

In quarternion notations these take the forms,

$$\left.\begin{aligned}
M_\sigma &= \tfrac{1}{6} S\sigma_1\sigma_2\sigma_3 S\nabla_1\nabla_2\nabla_3, \\
M_{Xx} &= \frac{\partial Y}{\partial y}\frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial z}\frac{\partial Z}{\partial y} = \tfrac{1}{2} Si\sigma_1\sigma_2\, Si\nabla_1\nabla_2 \;, \\
M_{Xy} &= \frac{\partial Y}{\partial z}\frac{\partial Z}{\partial x} - \frac{\partial Y}{\partial x}\frac{\partial Z}{\partial z} = \tfrac{1}{2} Si\sigma_1\sigma_2\, Sj\nabla_1\nabla_2 \;, \\
M_{Xz} &= \frac{\partial Y}{\partial x}\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial y}\frac{\partial Z}{\partial x} = \tfrac{1}{2} Si\sigma_1\sigma_2\, Sk\nabla_1\nabla_2 \;, \\
M_{Yx} &= \frac{\partial Z}{\partial y}\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial z}\frac{\partial X}{\partial y} = \tfrac{1}{2} Sj\sigma_1\sigma_2\, Si\nabla_1\nabla_2 \;, \\
M_{Yy} &= \frac{\partial Z}{\partial z}\frac{\partial X}{\partial x} - \frac{\partial Z}{\partial x}\frac{\partial X}{\partial z} = \tfrac{1}{2} Sj\sigma_1\sigma_2\, Sj\nabla_1\nabla_2 \;, \\
M_{Yz} &= \frac{\partial Z}{\partial x}\frac{\partial X}{\partial y} - \frac{\partial Z}{\partial y}\frac{\partial X}{\partial x} = \tfrac{1}{2} Sj\sigma_1\sigma_2\, Sk\nabla_1\nabla_2 \;, \\
M_{Zx} &= \frac{\partial X}{\partial y}\frac{\partial Y}{\partial z} - \frac{\partial X}{\partial z}\frac{\partial Y}{\partial y} = \tfrac{1}{2} Sk\sigma_1\sigma_2\, Si\nabla_1\nabla_2 \;, \\
M_{Zy} &= \frac{\partial X}{\partial z}\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial x}\frac{\partial Y}{\partial z} = \tfrac{1}{2} Sk\sigma_1\sigma_2\, Sj\nabla_1\nabla_2 \;, \\
M_{Zz} &= \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x} = \tfrac{1}{2} Sk\sigma_1\sigma^2\, Sk\nabla_1\nabla_2 \;.
\end{aligned}\right\} \tag{24}$$

Thus we may observe that there are at most nine independent quantities, hence these can be taken together in a linear vector function. By employing these quaternion forms in the expression,

$$\begin{aligned}
{}_\sigma\nabla &= i\frac{\partial}{\partial X} + j\frac{\partial}{\partial Y} + k\frac{\partial}{\partial Z} \\
&= \Sigma\, i\left[\frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial}{\partial y} + \frac{\partial z}{\partial X}\frac{\partial}{\partial z}\right],
\end{aligned}$$

we obtain the same results as in (18). The same kind of treatment applies to quaternion transformations.

§ 6.

Expressions of partial differential coefficients in terms of φ of § 3 lead to many interesting results.

Since by (12) and (13) we obtain

$$\left.\begin{aligned} \frac{\partial}{\partial X} &= -Si_\sigma\nabla = -S_\rho\nabla\varphi^{-1}i\,, \\ \frac{\partial}{\partial Y} &= -Sj_\sigma\nabla = -S_\rho\nabla\varphi^{-1}j\,, \\ \frac{\partial}{\partial Z} &= -Sk_\sigma\nabla = -S_\rho\nabla\varphi^{-1}k\,; \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= -Si_\rho\nabla = -S_\sigma\nabla\varphi i\,, \\ \frac{\partial}{\partial y} &= -Sj_\rho\nabla = -S_\sigma\nabla\varphi j\,, \\ \frac{\partial}{\partial z} &= -Sk_\rho\nabla = -S_\sigma\nabla\varphi k\,; \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} \frac{\partial}{\partial T} &= \quad S_p\nabla = \quad S\Phi'^{-1}{}_q\nabla\,, \\ \frac{\partial}{\partial X} &= -Si_p\nabla = -S_q\nabla\Phi^{-1}i\,, \\ \frac{\partial}{\partial Y} &= -Sj_p\nabla = -S_q\nabla\Phi^{-1}j\,, \\ \frac{\partial}{\partial Z} &= -Sk_p\nabla = -S_q\nabla\Phi^{-1}k\,; \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} \frac{\partial}{\partial t} &= \quad S_q\nabla = \quad S\Phi'_p\nabla\,, \\ \frac{\partial}{\partial x} &= -Si_q\nabla = -S_p\nabla\Phi i\,, \\ \frac{\partial}{\partial y} &= -Sj_q\nabla = -S_p\nabla\Phi j\,, \\ \frac{\partial}{\partial z} &= -Sk_q\nabla = -S_p\nabla\Phi k\,; \end{aligned}\right\} \tag{28}$$

we obtain for the transformation of vector variables,

$$\left.\begin{aligned}
&\frac{\partial x}{\partial X}=-Si\varphi^{-1}i\,, \quad \frac{\partial y}{\partial X}=-Sj\varphi^{-1}i\,, \quad \frac{\partial z}{\partial X}=-Sk\varphi^{-1}i\,;\\
&\frac{\partial x}{\partial Y}=-Si\varphi^{-1}j\,, \quad \frac{\partial y}{\partial Y}=-Sj\varphi^{-1}j\,, \quad \frac{\partial z}{\partial Y}=-Sk\varphi^{-1}j\,;\\
&\frac{\partial x}{\partial Z}=-Si\varphi^{-1}k\,, \quad \frac{\partial y}{\partial Z}=-Sj\varphi^{-1}k, \quad \frac{\partial z}{\partial Z}=-Sk\varphi^{-1}k;
\end{aligned}\right\} \qquad (29)$$

$$\left.\begin{aligned}
&\frac{\partial X}{\partial x}=-Si\varphi i\,, \quad \frac{\partial Y}{\partial x}=-Sj\varphi i\,, \quad \frac{\partial Z}{\partial x}=-Sk\varphi i\;;\\
&\frac{\partial X}{\partial y}=-Si\varphi j\,, \quad \frac{\partial Y}{\partial y}=-Sj\varphi j\,, \quad \frac{\partial Z}{\partial y}=-Sk\varphi j\;;\\
&\frac{\partial X}{\partial z}=-Si\varphi k, \quad \frac{\partial Y}{\partial z}=-Sj\varphi k, \quad \frac{\partial Z}{\partial z}=-Sk\varphi k\,;
\end{aligned}\right\} \qquad (30)$$

and for the transformation of quaternion variables

$$\left.\begin{aligned}
&\frac{\partial t}{\partial T}=S\Phi'^{-1}1, \quad \frac{\partial x}{\partial T}=S\Phi'^{-1}i, \quad \frac{\partial y}{\partial T}=S\Phi'^{-1}j, \quad \frac{\partial z}{\partial T}=S\Phi'^{-1}k\,;\\
&\frac{\partial t}{\partial X}=-S\Phi^{-1}i\,, \quad \frac{\partial x}{\partial X}=-Si\Phi^{-1}i, \quad \frac{\partial y}{\partial X}=-Sj\Phi^{-1}i\,, \quad \frac{\partial z}{\partial X}=-Sk\Phi^{-1}i\,;\\
&\frac{\partial t}{\partial Y}=-S\Phi^{-1}j\,, \quad \frac{\partial x}{\partial Y}=-Si\Phi^{-1}j, \quad \frac{\partial y}{\partial Y}=-Sj\Phi^{-1}j\,, \quad \frac{\partial z}{\partial Y}=-Sk\Phi^{-1}j;\\
&\frac{\partial t}{\partial Z}=-S\Phi^{-1}k\,, \quad \frac{\partial x}{\partial Z}=-Si\Phi^{-1}k, \quad \frac{\partial y}{\partial Z}=-Sj\Phi^{-1}k, \quad \frac{\partial z}{\partial Z}=-Sk\Phi^{-1}k.
\end{aligned}\right\} \qquad (31)$$

$$\left.\begin{aligned}
&\frac{\partial T}{\partial t}=S\Phi'1, \quad \frac{\partial X}{\partial t}=S\Phi'i\,, \quad \frac{\partial Y}{\partial t}=S\Phi'j\,, \quad \frac{\partial Z}{\partial t}=S\Phi'k\,;\\
&\frac{\partial T}{\partial x}=-S\Phi i\,, \quad \frac{\partial X}{\partial x}=-Si\Phi i\,, \quad \frac{\partial Y}{\partial x}=-Sj\Phi i, \quad \frac{\partial Z}{\partial x}=-Sk\Phi i\,;\\
&\frac{\partial T}{\partial y}=-S\Phi j\,, \quad \frac{\partial X}{\partial y}=-Si\Phi j, \quad \frac{\partial Y}{\partial y}=-Sj\Phi j, \quad \frac{\partial Z}{\partial y}=-Sk\Phi j\,;\\
&\frac{\partial T}{\partial z}=-S\Phi k, \quad \frac{\partial X}{\partial z}=-Si\Phi k, \quad \frac{\partial Y}{\partial z}=-Sj\Phi k, \quad \frac{\partial Z}{\partial z}=-Sk\Phi k\,.
\end{aligned}\right\} \qquad (32)$$

Thus, when φ and Φ are self-conjugate functions, we obtain

$$\left.\begin{aligned} V_\sigma\nabla\rho &= 0\,, \\ V_\rho\nabla\sigma &= 0\,; \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} V_q\nabla p &= 0\,, \\ V_p\nabla q &= 0\,. \end{aligned}\right\} \tag{34}$$

Expressed in Cartesian symbols these become

$$\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}\,, \quad \frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}\,, \quad \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}\,;$$

$$\frac{\partial y}{\partial X} = \frac{\partial x}{\partial Y}\,, \quad \frac{\partial z}{\partial Y} = \frac{\partial y}{\partial Z}\,, \quad \frac{\partial x}{\partial Z} = \frac{\partial z}{\partial X}\,.$$

We shall notice here that the second equation of (33) and the first of (34) are respectively a condition of integrability of

$$\left.\begin{aligned} S\sigma d\rho &= 0\,,^* \\ Spdq &= 0\,, \end{aligned}\right\} \tag{35}$$

in the forms $f\rho = c$, $Fq = c$, respectively; and (33) and (34) show that when these are integrable then $S\rho d\sigma = 0$ and $Sqdp = 0$ are also integrable, which is also at once seen from the identities

$$dS\rho\sigma = S\rho d\sigma + S\sigma d\rho\,,$$

$$dSpq = Spdq + Sqdp\,.$$

Again, if φ, Φ are self-conjugate, we have for any constant vector α and any constant quaternion a,

$$\left.\begin{aligned} {}_\sigma\nabla Si\rho &= -\,\varphi^{-1}i\,, & {}_\rho\nabla Si\sigma &= -\,\varphi i\,, \\ {}_\sigma\nabla Sj\rho &= -\,\varphi^{-1}j\,, & {}_\rho\nabla Sj\sigma &= -\,\varphi j\,, \\ {}_\sigma\nabla Sk\rho &= -\,\varphi^{-1}k\,; & {}_\rho\nabla Sk\sigma &= -\,\varphi k\,; \\ {}_\sigma\nabla Sa\rho &= -\,\varphi^{-1}a\,: & {}_\rho\nabla Sa\sigma &= -\,\varphi a\,: \end{aligned}\right\} \tag{36}$$

* Tait's Quaternions, 3d ed., p. 253; Hamilton's Elements, Art. 415.

$$\left.\begin{aligned}
{}_p\nabla Sq &= K\eta S\Phi^{-1}\eta, & {}_q\nabla Sp &= K\eta S\Phi\eta\ , \\
{}_p\nabla Siq &= -\ \Phi^{-1}i\ , & {}_q\nabla Sip &= -\ \Phi i\ , \\
{}_p\nabla Sjq &= -\ \Phi^{-1}j\ , & {}_q\nabla Sjp &= -\ \Phi j\ , \\
{}_p\nabla Skq &= -\ \Phi^{-1}k\ ; & {}_q\nabla Skp &= -\ \Phi k\ ; \\
{}_p\nabla Saq &= \Phi^{-1}Ka\ : & {}_q\nabla Sap &= -\ \Phi Ka:
\end{aligned}\right\} \tag{37}$$

in which the η-pairs are extensions of ζ-pairs, such that

$$\left.\begin{aligned}
Q(\eta,\eta) &= Q({}_q\nabla_1, q_1) = Q(q_1, {}_q\nabla_1) \\
&= Q(1,1) + Q(i,i) + Q(j,j) + Q(k,k) \\
&= Q(1,1) + Q(\zeta,\zeta)\ ,
\end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned}
Q(K\eta,\eta) &= Q(\eta, K\eta) = Q({}_{Kq}\nabla_1, q_1) \\
&= Q(q_1, {}_{Kq}\nabla) = Q(Kq_1, {}_q\nabla) = Q({}_q\nabla, Kq_1) \\
&= Q(1,1) - Q(\zeta,\zeta)\ ;
\end{aligned}\right\} \tag{39}$$

where Q is any quaternion function linear in each of the arguments mentioned.

Now, whatever the linear vector function φ may be, we have

$$\left.\begin{aligned}
\frac{\partial\rho}{\partial X} &= \varphi^{-1}i\ , & \frac{\partial\sigma}{\partial x} &= \varphi i\ , \\
\frac{\partial\rho}{\partial Y} &= \varphi^{-1}j\ , & \frac{\partial\sigma}{\partial y} &= \varphi j\ , \\
\frac{\partial\rho}{\partial Z} &= \varphi^{-1}k\ . & \frac{\partial\sigma}{\partial z} &= \varphi k\ .
\end{aligned}\right\} \tag{40}$$

Hence, we obtain

$$\left.\begin{aligned}
{}_\sigma\nabla\rho &= \zeta\varphi^{-1}\zeta = \frac{1}{m}(2\delta - m')\ , \\
S_\sigma\nabla\rho &= -\frac{m'}{m}, \quad V_\sigma\nabla\rho = \frac{2}{m}\delta;
\end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned}
{}_\rho\nabla\sigma &= \zeta\varphi\zeta = 2\gamma - m''\ , \\
S_\rho\nabla\sigma &= -\ m'', \quad V_\rho\nabla\sigma = 2\gamma\ ;
\end{aligned}\right\} \tag{42}$$

where m, m', m'' are the three invariants of φ, viz.:

$$\varphi^3 - m''\varphi^2 + m'\varphi - m = 0\ ,$$

and γ, δ are rotation vectors of φ and its Hamiltonian inverse function ψ; thus

$$\left.\begin{aligned} m &= -\tfrac{1}{3}S\varphi'\zeta\psi\zeta\,, & m' &= -S\zeta\psi\zeta\,, & m'' &= -S\zeta\varphi\zeta\,; \\ 2\gamma &= V\zeta\varphi\zeta\,, & 2\delta &= V\zeta\psi\zeta\,, & \delta &= \varphi\gamma\,. \end{aligned}\right\} \tag{43}$$

Similarly whatever the linear quaternion function Φ may be, we have

$$\left.\begin{aligned} \frac{\partial q}{\partial T} &= \eta S\Phi'^{-1}\eta\,, & \frac{\partial p}{\partial t} &= \eta S\Phi'\eta\,, \\ \frac{\partial q}{\partial X} &= -K\Phi^{-1}i\,, & \frac{\partial p}{\partial x} &= -K\Phi i\,, \\ \frac{\partial q}{\partial Y} &= -K\Phi^{-1}j\,, & \frac{\partial p}{\partial y} &= -K\Phi j\,, \\ \frac{\partial q}{\partial Z} &= -K\Phi^{-1}k\,; & \frac{\partial p}{\partial z} &= -K\Phi k\,. \end{aligned}\right\} \tag{45}$$

Thus we obtain

$$\left.\begin{aligned} {}_p\nabla q &= \eta S\Phi'^{-1}\eta - \zeta K\Phi^{-1}\zeta, \\ S_p\nabla q &= S\eta\Phi^{-1}\eta \\ V_p\nabla q &= \zeta S(\Phi'^{-1} - \Phi^{-1})\zeta + V\zeta V\Phi^{-1}\zeta\,. \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} {}_q\nabla p &= \eta S\Phi'\eta - \zeta K\Phi\zeta, \\ S_q\nabla p &= S\eta\Phi\eta\,, \\ V_q\nabla p &= \zeta S(\Phi' - \Phi)\zeta + V\zeta V\Phi\zeta\,. \end{aligned}\right\} \tag{47}$$

Thus by (34), (46), (47) we have for any self-conjugate linear quaternion function Φ,

$$V\zeta V\Phi\zeta = 0\,, \quad V\zeta V\Phi^{-1}\zeta = 0\,. \tag{48}$$

This means also, that when Φ is self-conjugate, the linear vector function φ derived* from it is also self-conjugate, where φ is defined for any vector τ and any quaternion s, by

$$\left.\begin{aligned} \varphi Vs &= V\Phi Vs \\ \text{or} \qquad \varphi\tau &= V\Phi\tau\,. \end{aligned}\right\} \tag{49}$$

Hence in (31) and (32), if α, β stand for either of the unit vectors i, j, k we have

$$S\alpha\Phi\beta = S\alpha\varphi\beta\,, \quad S\alpha\Phi^{-1}\beta = S\alpha\varphi^{-1}\beta\,. \tag{50}$$

* In regard to the reduction of Φ into φ, see Hamilton's Elements, Art. 364.

§ 7.

In the definitions of ρ, q, ${}_{\rho}\nabla$, ${}_{q}\nabla$, if we put

$$\rho_1 = ix\,, \quad \rho_2 = jy\,, \quad \rho_3 = kz\,;$$

$$q_1 = t\,, \quad q_2 = ix\,, \quad q_3 = jy\,, \quad q_4 = kz\,;$$

and further take t, x, y, z to be four independent variables, then the ρ's and q's are entirely independent of each other both in tensors and in versors; and we have

$${}_{\rho}\nabla = {}_{\Sigma\rho_1}\nabla = \Sigma_{\rho_1}\nabla\,, \quad {}_{q}\nabla = {}_{\Sigma q_1}\nabla = \Sigma_{q_1}\nabla\,.$$

Consistent with this, if ρ, σ, τ, . . . are independent vector variables, and if p, q, r, . . . are independent quaternion variables, we may write

$${}_{\Sigma\rho}\nabla = \Sigma_{\rho}\nabla\,, \qquad {}_{\Sigma q}\nabla = \Sigma_{q}\nabla\,. \tag{51}$$

However, although we may write

$${}_{\rho_1}\nabla = {}_{ix}\nabla = i\,\frac{\partial}{\partial x} = U\rho_1\,\frac{\partial}{\partial T\rho_1}\,,$$

we cannot extend this into

$${}_{\rho}\nabla = U\rho\,\frac{\partial}{\partial T\rho}\,, \text{ or } {}_{q}\nabla = Uq\,\frac{\partial}{\partial Tq}\,,$$

since in the first case $U\rho_1$ is a constant vector, while in the second $U\rho$ or Uq is a variable vector or quaternion. Now to prove (51) we have merely to write

$$d = SdK\Sigma\rho\,{}_{\Sigma\rho}\nabla = \Sigma SdK\rho\,{}_{\rho}\nabla = SdK\Sigma\rho\Sigma_{\rho}\nabla\,,$$

and

$$d = SdK\Sigma q\,{}_{\Sigma q}\nabla = \Sigma SdKq\,{}_{q}\nabla = SdK\Sigma q\Sigma_{q}\nabla\,,$$

in which, on account of the independency of the constituents, in differentiating any function Q, such terms as

$$SdK\rho\,{}_{\sigma}\nabla \text{ or } SdKq\,{}_{p}\nabla$$

do not exist.

However, when the ρ's or q's are functions of one of themselves, or when they are functions of some other vector or quaternion, such simple results as (51) cannot be obtained; for, since in this case

$$d\Sigma\rho = (\Sigma\varphi)\,d\rho\,,$$

$$d\Sigma q = (\Sigma\Phi)\,dq\,,$$

we have

$$\left.\begin{array}{l} {}_{\Sigma\rho}\nabla = (\Sigma\varphi)'^{-1}{}_{\rho}\nabla\,, \text{ or } {}_{\rho}\nabla = (\Sigma\varphi)'\,{}_{\Sigma\rho}\nabla\,; \\ {}_{\Sigma q}\nabla = (\Sigma\Phi)'^{-1}{}_{q}\nabla\,, \text{ or } {}_{q}\nabla = (\Sigma\Phi)'{}_{\Sigma q}\nabla. \end{array}\right\} \tag{52}$$

By the definitions (A), (B), (C), whether t, x, y, z are independent or not, we have

$$\left.\begin{array}{rcrcr} {}_{Kt}\nabla t &=& {}_{t}\nabla Kt &=& 1\,, \\ {}_{K\rho}\nabla\rho &=& {}_{\rho}\nabla K\rho &=& 3\,, \\ {}_{Kq}\nabla q &=& {}_{q}\nabla Kq &=& 4\,, \\ {}_{q}\nabla q &=& {}_{Kq}\nabla Kq &=& -\,2\,. \end{array}\right\} \tag{53}$$

But, if we take n quantities, ρ, σ, τ, . . . , and n other quantities, p, q, r. . . . , to be independent vectors and quaternions, we have

$$\left.\begin{array}{rcrcr} {}_{\Sigma t}\nabla\Sigma t &=& \Sigma 1 &=& n\,, \\ {}_{\Sigma\rho}\nabla\Sigma K\rho &=& \Sigma 3 &=& 3n\,, \\ {}_{\Sigma q}\nabla\Sigma Kq &=& \Sigma 4 &=& 4n\,, \\ {}_{\Sigma q}\nabla\Sigma q &=& -\,\Sigma 2 &=& -\,2n\,. \end{array}\right\} \tag{54}$$

Since, however, whether the constituents are independent or not, we may write

$$\Sigma t = T, \quad \Sigma\rho = P, \quad \Sigma q = Q\,;$$

we have also

$$\left.\begin{array}{rcrcr} {}_{\Sigma t}\nabla\Sigma t &=& {}_{T}\nabla T &=& 1, \\ {}_{\Sigma\rho}\nabla\Sigma K\rho &=& {}_{P}\nabla KP &=& 3, \\ {}_{\Sigma q}\nabla\Sigma Kq &=& {}_{Q}\nabla KQ &=& 4, \\ {}_{\Sigma q}\nabla\Sigma q &=& {}_{Q}\nabla Q &=& -\,2. \end{array}\right\} \tag{55}$$

Hence we must distinguish between the two cases, in the first of which the constituents of the arguments are taken separately, while in the latter they are taken conjointly, and for the latter case I propose the notations,

$${}_{(\Sigma t)}\nabla\,, \quad {}_{(\Sigma\rho)}\nabla\,, \quad {}_{(\Sigma q)}\nabla\,. \tag{56}$$

In connection with this, I might mention that when we take nine scalars of any linear vector function φ as independent variables, then, as McAulay* has shown,

$$\delta Q = -\,Q_1\,S\delta\varphi\zeta\,{}_{\phi}\Pi_1\,\zeta\,.$$

* Utility of Quaternions, etc., Art. 5.

This may also be written

$$
\begin{aligned}
\delta Q &= -\,Q_1\,S\delta\varphi\zeta\,{}_{\phi\zeta}\nabla_1 \\
&= -\,Q_1\,S\delta\varphi i\,{}_{\phi i}\nabla_1 - Q_1\,S\delta\varphi j\,{}_{\phi j}\nabla_1 - Q_1\,S\delta\varphi k\,{}_{\phi k}\nabla_1\,.
\end{aligned}
$$

That is, the total variation of Q is equal to the sum of the variations of the same due to the variations of three vectors φi, φj, φk which are independent of each other.

§ 8.

Seeing that the nabla of a quaternion argument is capable of direct physical application, I propose now to transform such a nabla into the same operator with a vector for its argument, and vice versa.

The simplest case is when the vector element of the quaternion argument is the vector under consideration, thus

$$
{}_q\nabla = {}_{S_q}\nabla + {}_{V_q}\nabla = {}_{S_q}\nabla + {}_\rho\nabla\,. \tag{57}
$$

In general, let p be a quaternion function of a vector ρ, and let σ be a vector function of a quaternion q; then we have

$$
\left.\begin{aligned}
dp &= \Phi d\rho\,, \\
d\sigma &= V\Psi dq\,;
\end{aligned}\right\} \tag{58}
$$

where Φ, Ψ are linear quaternion functions, having no relations between them. We cannot produce the second from the first, nor the first from the second; for $\Phi 1$ is not given in the first case, nor $S\Psi dq$ in the second.

Thus,

$$
\begin{aligned}
d &= Sdp\,{}_{Kp}\nabla = S\Phi d\rho\,{}_{Kp}\nabla = Sd\rho\Phi'\,{}_{Kp}\nabla = Sd\rho\,{}_{K\rho}\nabla \\
d &= Sd\sigma\,{}_{K\sigma}\nabla = S\Psi dq\,{}_{K\sigma}\nabla = Sdq\Psi'\,{}_{K\sigma}\nabla = Sdq\,{}_{Kq}\nabla\,;
\end{aligned}
$$

whence

$$
\left.\begin{aligned}
{}_{K\rho}\nabla &= V\Phi'\,{}_{Kp}\nabla\,, \quad \text{or} \quad {}_\rho\nabla = -\,V\Phi' K\,{}_p\nabla\,; \\
{}_{Kq}\nabla &= \Psi'\,{}_{K\sigma}\nabla\,, \quad \text{or} \quad {}_q\nabla = -\,K\Psi'\,{}_\sigma\nabla\,;
\end{aligned}\right\} \tag{59}
$$

or, also,

$$
{}_\rho\nabla = {}_\rho\nabla_1\,Sp_1\,{}_{Kp}\nabla\,, \quad {}_q\nabla = {}_q\nabla_1\,S\sigma_1\,{}_{K\sigma}\nabla\,. \tag{59'}
$$

Thus Laplace's operators become

$$
\left.\begin{aligned}
S\,{}_\rho\nabla\,{}_{K\rho}\nabla &= SV\Phi' K\,{}_p\nabla\,V\Phi' K\,{}_p\nabla = -\,SV\Phi'\,{}_{Kp}\nabla\,V\Phi'\,{}_{Kp}\nabla\,, \\
S\,{}_q\nabla\,{}_{Kq}\nabla &= S\Psi'\,{}_\sigma\nabla K\Psi\,{}_\sigma\nabla = S\,{}_\sigma\nabla\,\Psi K\Psi'\,{}_\sigma\nabla\,, \\
S\,{}_q\nabla\,{}_q\nabla &= S\,{}_{Kq}\nabla\,{}_{Kq}\nabla = S\Psi'\,{}_\sigma\nabla\,\Psi'\,{}_\sigma\nabla = S\,{}_\sigma\nabla\,\Psi\Psi'\,{}_\sigma\nabla\,.
\end{aligned}\right\} \tag{60}
$$

Since the general form of a linear quaternion function is

$$\left.\begin{aligned} &\Phi q = \Sigma a' S a q = a' S a q + b' S b q + c' S c q + d' S d q\,, \\ \text{we have}\qquad &\Phi' q = \Sigma a S a' q\,, \qquad K\Phi q = \Sigma K a' S a q\,, \\ &K\Phi' q = \Sigma K a S a' q\,, \qquad \Phi K q = \Sigma a' S a K q\,, \\ &(K\Phi)' q = \Sigma a S K a' q\,, \qquad K\Phi q = \Sigma K a' S a q\,, \end{aligned}\right\} \tag{61}$$

and so on.

Thus when q, σ are functions of t only, we write ${}_t\nabla$ for $\dfrac{d}{dt}$ since it is the sole independent scalar variable ; then

$${}_t\nabla = {}_t\nabla_1 S\sigma_1\,{}_{K\sigma}\nabla = {}_t\nabla_1 S\Psi q_1\,{}_{K\sigma}\nabla = {}_t\nabla_1 S q_1 \Psi'\,{}_{K\sigma}\nabla = {}_t\nabla_1 S q_1\,{}_{Kq}\nabla.$$

Again, when p and ρ are functions of t only,

$${}_t\nabla = {}_t\nabla_1 S p_1\,{}_{Kp}\nabla = {}_t\nabla_1 S\Psi\rho_1\,{}_{Kp}\nabla = {}_t\nabla_1 S\rho_1 V\Psi'\,{}_{Kp}\nabla = {}_t\nabla_1 S\rho_1\,{}_{K\rho}\nabla\,.$$

These results agree with those obtained before.

§ 9.

We shall now consider the more general transformation of nablas, viz. when the arguments are expressed in variable scalars and vectors which have certain geometrical relations between themselves.

Take σ to be a vector whose form is given by

$$\sigma = l\lambda + m\mu + n\nu\,, \tag{62}$$

and write

$${}_\sigma\nabla = \lambda\frac{\partial}{\partial l} + \mu\frac{\partial}{\partial m} + \nu\frac{\partial}{\partial n}\,, \tag{63}$$

where

$$\lambda\mu = \nu\,, \quad \mu\nu = \lambda\,, \quad \nu\lambda = \mu\,. \tag{64}$$

Thus, let λ, μ, ν be a rectangular system of unit vectors in which l, m, n are all functions of a position vector ρ which may be written in the form,

$$\rho = ix + jy + kz\,,$$

where i, j, k are as usual a rectangular unit system, being constant through all operations, although they are arbitrary at first choice. Since then ρ is the independent variable vector we may write at any time,

$$\begin{aligned} {}_\rho\nabla &= i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \\ &= i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\,. \end{aligned} \tag{65}$$

Then, if we take a linear vector function of the form

$$\pi\tau = \nabla l S\lambda\tau + \nabla m S\mu\tau + \nabla n S\nu\tau\,, \tag{66}$$

where ∇ is used for ${}_\rho\nabla$ when it is already applied; we have the results

$$\left.\begin{aligned} {}_\rho\nabla &= -\pi{}_\sigma\nabla\,, \\ {}_\sigma\nabla &= -\pi^{-1}{}_\rho\nabla\,. \end{aligned}\right\} \tag{67}$$

Laplace's operator becomes

$$S_\rho\nabla K_\rho\nabla = S_\sigma\nabla\pi\pi' K_\sigma\nabla\,, \tag{68}$$

where $\pi\pi' = \pi'\pi$ is a self-conjugate linear vector function.

To verify the results (67) we obtain, by using them,

$$\left.\begin{aligned} \nabla l &= -\pi_\sigma\nabla l = -\pi({}_\sigma\nabla l) = -\pi\lambda\,, \\ \nabla m &= -\pi_\sigma\nabla m = -\pi({}_\sigma\nabla m) = -\pi\mu\,, \\ \nabla n &= -\pi_\sigma\nabla n = -\pi({}_\sigma\nabla n) = -\pi\nu\,, \end{aligned}\right\} \tag{69}$$

which are obtained at once from the definition of π.

Again, since we have the inverse function in the form

$$-\pi^{-1}\tau = \lambda S\nabla m\nabla n\tau + \mu S\nabla n\nabla l\tau + \nu S\nabla l\nabla m\tau : S\nabla l\nabla m\nabla n\,, \tag{70}$$

we obtain by assuming the result of (67),

$$\left.\begin{aligned} {}_\sigma\nabla l &= -\pi^{-1}{}_\rho\nabla l = -\pi^{-1}(\nabla l) = -\lambda\,, \\ {}_\sigma\nabla m &= -\pi^{-1}{}_\rho\nabla m = -\pi^{-1}(\nabla m) = -\mu\,, \\ {}_\sigma\nabla n &= -\pi^{-1}{}_\rho\nabla n = -\pi^{-1}(\nabla n) = -\nu\,, \end{aligned}\right\} \tag{71}$$

which are also known from the definition of ${}_\sigma\nabla$.

This transformation is useful in considering the orthogonal system of surfaces. Thus, if we have for these surfaces the equations

$$F_1(\rho) = c_1\,, \quad F_2(\rho) = c_2\,, \quad F_3(\rho) = c_3\,, \tag{73}$$

then the conditions of orthogonality are

$$\left.\begin{aligned} S\nabla F_1\nabla F_2 &= 0\,, \\ S\nabla F_2\nabla F_3 &= 0\,, \\ S\nabla F_3\nabla F_1 &= 0\,. \end{aligned}\right\} \tag{74}$$

Since we may write

$$\left.\begin{aligned}\nabla F_1 &= l\lambda\ ,\\ \nabla F_2 &= m\mu,\\ \nabla F_3 &= n\nu\ ,\end{aligned}\right\} \qquad (75)$$

the form of π is given by

$$\pi\tau = \nabla \log T\nabla F_1 S\nabla F_1\tau + \nabla \log T\nabla F_2 S\nabla F_2\tau + \nabla \log T\nabla F_3 S\nabla F_3\tau\ . \quad (76)$$

10.

Perhaps it might be well to remark on the differential and variation symbols as they are usually expressed by quaternion notations. Thus, we write

$$\left.\begin{aligned}\delta &= -S\delta\sigma_\sigma\nabla\ ,\\ d &= -Sd\sigma_\sigma\nabla\ .\end{aligned}\right\} \qquad (77)$$

Here $\delta\sigma$ and $d\sigma$ are purely arbitrary vectors, they may take any values either in direction or in length, any variations in the values of $\delta\sigma$ and $d\sigma$ giving rise to corresponding variations in the values of δ and d on the left side.

For the sake of definiteness I propose to call δ, as it is written in (77), an *unconditional* or *total variation operator*, and $\delta\sigma$ an *unconditional* or *total variation*, and d and $d\sigma$, respectively, an *unconditional* or *total differential operator* and *differential;* on the other hand, in

$$\left.\begin{aligned}\delta_{\|\alpha} &= -S\delta_{\|\alpha}\sigma_\sigma\nabla\ ,\\ \delta_{\|\alpha} &= -Sd_{\|\alpha}\sigma_\sigma\nabla\ ,\end{aligned}\right\} \qquad (78)^*$$

$\delta_{\|\alpha}\sigma$ and $d_{\|\alpha}\sigma$ are used to indicate that the vector σ varies in a given direction parallel to α, and here the words *conditional* or *partial* are substituted for the words *unconditional* or *total* of the former cases.

Since the conditional variation is the component of the unconditional variation in the direction of α, we have

$$\left.\begin{aligned}\delta_{\|\alpha}\sigma &= \alpha^{-1}S\alpha\delta\sigma = \alpha S\alpha^{-1}\delta\sigma\ ,\\ \delta_{\|\alpha}\sigma &= \alpha^{-1}S\alpha d\sigma = \alpha S\alpha^{-1}d\sigma\ .\end{aligned}\right\} \qquad (79)$$

Hence we obtain conditional variation and differential operators in terms of unconditional ones,

$$\left.\begin{aligned}\delta_{\|\alpha} &= -S\alpha^{-1}{}_\sigma\nabla S\alpha\delta\sigma = S\,V\delta\sigma\alpha^{-1}\,V\alpha_\sigma\nabla + \delta\ ,\\ d_{\|\alpha} &= -S\alpha^{-1}{}_\sigma\nabla S\alpha d\sigma = S\,Vd\sigma\alpha^{-1}\,V\alpha_\sigma\nabla + d\ .\end{aligned}\right\} \qquad (80)$$

* Here I add the sign of parallelism, in order to distinguish the similar operator d_α in Tait's Quaternions, 3rd ed., p. 372.

Thus, in the case of orthogonal surfaces mentioned in the last section, writing

$$\left.\begin{aligned} \nabla F_1 &= l\lambda \;= \Lambda\,, \\ \nabla F_2 &= m\mu = M\,, \\ \nabla F_3 &= n\nu \;= N\,, \end{aligned}\right\} \tag{81}$$

Λ, M, N are normals to the surfaces, and on account of orthogonality

$$\Lambda \parallel d_{23}\rho\,, \quad M \parallel d_{31}\rho\,, \quad N \parallel d_{12}\rho\,, \tag{82}$$

where, for instance, $d_{12}\rho$ denotes the tangent to the curve of intersection of the surfaces F_1 and F_2 at the point ρ, or it is the change of the position vector in the direction of this curve of intersection. Hence we have to distinguish the three cases of the variation for each of the three normals according as the position vector changes along the curve 12, or 23, or 31; hence the importance of notation like (78).

Thus, instead of

$$d\Lambda = -\,\Lambda_1 Sd\rho_{\rho}\nabla_1\,,$$

we have

$$\left.\begin{aligned} d_{\parallel 12}\Lambda = d_{\parallel N}\Lambda &= -\,\Lambda_1 Sd_{\parallel 12}\rho\,_{\rho}\nabla_1 &&= -\,\Lambda_1 Sd_{\parallel N}\rho\,_{\rho}\nabla_1\,, \\ &= -\,\Lambda_1 SN^{-1}{}_{\rho}\nabla_1 SNd\rho &&= \Lambda_1 SVd\rho N^{-1}VN_{\rho}\nabla_1 + d\Lambda\,, \\ d_{\parallel 23}\Lambda = d_{\parallel \Lambda}\Lambda &= -\,\Lambda_1 Sd_{\parallel 23}\rho\,_{\rho}\nabla_1 &&= -\,\Lambda_1 Sd_{\parallel \Lambda}\rho\,_{\rho}\nabla_1\,, \\ &= -\,\Lambda_1 S\Lambda^{-1}{}_{\rho}\nabla_1 S\Lambda d\rho &&= \Lambda_1 SVd\rho\Lambda^{-1}V\Lambda_{\rho}\nabla_1 + d\Lambda\,, \\ d_{\parallel 31}\Lambda = d_{M}\Lambda &= -\,\Lambda_1 Sd_{\parallel 31}\rho\,_{\rho}\nabla_1 &&= -\,\Lambda_1 Sd_{\parallel M}\rho\,_{\rho}\nabla_1\,, \\ &= -\,\Lambda_1 SM^{-1}{}_{\rho}\nabla_1 SMd\rho &&= \Lambda_1 SVd\rho M^{-1}VM_{\rho}\nabla_1 + d\Lambda\,; \end{aligned}\right\} \tag{83}$$

and similarly for M and N.

Thus, for any function Ω, we have

$$\begin{aligned} d\Omega &= d_{\parallel\Lambda}\Omega + d_{\parallel M}\Omega + d_{\parallel N}\Omega \\ &= d_{\parallel i}\Omega + d_{\parallel j}\Omega + d_{\parallel k}\Omega\,; \end{aligned}$$

or, symbolically,

$$d = d_{\parallel\lambda} + d_{\parallel\mu} + d_{\parallel\nu}\,, \tag{84}$$

where λ, μ, ν constitute any rectangular system of either constant or variable unit vectors, or not; for

$$\begin{aligned} d_{\parallel\lambda} + d_{\parallel\mu} + d_{\parallel\nu} &= -\,S_{\rho}\nabla\lambda^{-1}S\lambda d\rho - S_{\rho}\nabla\mu^{-1}S\mu d\rho - S_{\rho}\nabla\nu^{-1}S\nu d\rho \\ &= -\,S_{\rho}\nabla d\rho = d\,. \end{aligned}$$

Thus the sum of conditional differentials or variations in three rectangular directions is equal to the unconditional differential or variation.

Again, if we denote by ${}_\rho\nabla_{\parallel\sigma}$ the component of ${}_\rho\nabla$ in the direction of σ, in the same manner that $d_{\parallel\sigma}\rho$ has been used to denote the component of $d\rho$ in the same direction, that is, if

$$
{}_\rho\nabla_{\parallel\sigma} = \sigma^{-1} S\sigma{}_\rho\nabla = \sigma S\sigma^{-1}{}_\rho\nabla\,, \tag{85}
$$

we may extend the definitions of (78), viz.,

$$
\left.\begin{aligned}
\delta_{\parallel\sigma} &= -\, S d_{\parallel\sigma}\rho\,{}_\rho\nabla = -\, S\delta\rho\,{}_\rho\nabla_{\parallel\sigma}\,,\\
d_{\parallel\sigma} &= -\, S d_{\parallel\sigma}\rho\,{}_\rho\nabla = -\, S d\rho\,{}_\rho\nabla_{\parallel\sigma}\,,
\end{aligned}\right\} \tag{86}
$$

where σ may be any vector either constant or variable ; and here we must have the names *unconditional* and *conditional nablas.*

But when the independent vector variable is required to move in a given direction τ, we do not have *necessarily*

$$
d_{\parallel\tau} = -\, S d_{\parallel\tau}\rho\,{}_\rho\nabla = -\, S d_{\parallel\tau}\sigma\,{}_\sigma\nabla\,.
$$

If we define Q to be a linear function of the argument ζ,

$$
\begin{aligned}
Q(\zeta) &= Q(i) + Q(j) + Q(k)\\
&= Q(\lambda) + Q(\mu) + Q(\nu)\,,
\end{aligned} \tag{87}
$$

where λ, μ, ν form any variable unit rectangular system, then for any vector σ,

with

$$
\left.\begin{aligned}
d_\zeta\sigma &= d\sigma\\
d_{\parallel\zeta}\sigma &= d_\zeta\sigma\\
d_{\parallel\zeta} &= d\,.
\end{aligned}\right\} \tag{88}
$$

Again, if

$$
d\sigma = \varphi d\rho\,,
$$

$$
d\rho = \varphi^{-1} d\sigma\,,
$$

then

$$
\left.\begin{aligned}
&{}_\sigma\nabla_{\parallel\zeta} = {}_\sigma\nabla\,, && {}_\rho\nabla_{\parallel\zeta} = {}_\rho\nabla\,;\\
&{}_\sigma\nabla_{\parallel\zeta} = \varphi'^{-1}{}_\rho\nabla_{\parallel\zeta}\,, && {}_\rho\nabla_{\parallel\zeta} = \varphi'{}_\sigma\nabla_{\parallel\zeta}\,;\\
&d_{\parallel\zeta}\sigma = \varphi d_{\parallel\zeta}\rho\,, && d_{\parallel\zeta}\rho = \varphi^{-1} d_{\parallel\zeta}\sigma.
\end{aligned}\right\} \tag{89}
$$

All these are old things in new dress, and so need no demonstration.

An unconditional variation, differential, or nabla is the *maximum* variation, differential, or nabla, inasmuch as the conditional ones are components. Thus, if Ω be the force, W the work done by the force, $d\rho$ the element of the path, σ the function of ρ being a given curve, then for the maximum work done, we have

$$dW = -S\Omega d\rho = -W_1 S_\rho \nabla_1 d\rho\,.$$

However, when the path is constrained to be on a curve whose tangent is σ,

$$d_{\|\sigma} W = -S\Omega d_{\|\sigma}\rho = -S\Omega_{\|\sigma} d\rho\,,$$

where $\Omega_{\|\sigma}$ is the component of the force along the tangent of the curve σ. If it happens that the force has a potential P, such that

$$\Omega = {}_\rho\nabla P\,,$$

then

$$d_{\|\sigma} W = -W_1 S_\rho \nabla_1 d_{\|\sigma}\rho\,,$$

$$= -S\Omega\, d_{\|\sigma}\rho = -P_1 S_\rho \nabla_1 d_{\|\sigma}\rho\,.$$

Hence, whatever may be the curve σ, we have

$$W_2 - W_1 = P_2 - P_1\,.$$

As another illustration, we may prove the following theorem :—Any system of three surfaces which cut each other in their lines of curvatures, cut each other in constant angles. Let ν_1, ν_2, ν_3 be normals to the three surfaces ; let $d_{12}\rho$, $d_{23}\rho$, $d_{31}\rho$, be the tangents to the curves of intersection of the surfaces 1, 2 ; 2, 3 ; 3, 1, respectively ; then the conditions that the surfaces 1, 2 cut each other in their lines of curvatures are

$$0 = S\nu_1 d\nu_1 d_{\|12}\rho = SV\nu_1 d\nu_1\, V\nu_1\nu_2 = S\nu_1\nu_2 S\nu_1 d\nu_1 - \nu_1^2 S\nu_2 d\nu_1\,,$$

$$0 = S\nu_2 d\nu_2 d_{\|12}\rho = SV\nu_2 d\nu_2\, V\nu_1\nu_2 = S\nu_1\nu_2 S\nu_2 d\nu_2 - \nu_2^2 S\nu_1 d\nu_2\,;$$

whence we obtain

$$S\nu_2 d\nu_1 = \frac{1}{\nu_1^2} S\nu_1 d\nu_1 S\nu_1\nu_2\,,$$

$$S\nu_1 d\nu_2 = \frac{1}{\nu_2^2} S\nu_2 d\nu_2 S\nu_1\nu_2\,.$$

Adding these, we obtain

$$\frac{dS\nu_1\nu_2}{S\nu_1\nu_2} = \tfrac{1}{2}\left[\frac{d\nu_1^2}{\nu_1^2} + \frac{d\nu_2^2}{\nu_2^2}\right],$$

or,

$$d \log S\nu_1\nu_2 = \tfrac{1}{2} d \log(\nu_1^2\nu_2^2) = \mathrm{d}\log(T\nu_1 T\nu_2)\,.$$

Integrating, we have

$$S\nu_1\nu_2 = -c_3 T\nu_1\nu_2 .$$

whence

$$\cos(\nu_1, \nu_2) = c_3 ,$$

and similarly,

$$\cos(\nu_2, \nu_3) = c_1 ,$$

$$\cos(\nu_3, \nu_1) = c_2 .$$

Of course this system of three surfaces includes the orthogonal system of surfaces.

§ 11.

For any function Q, linear in each of the arguments, McAulay* defined his very ingenious ζ-pairs in the form

$$Q(\zeta, \zeta) = Q({}_\rho\nabla_1, \rho_1) = Q(\rho_1, {}_\rho\nabla_1) ,$$

and thus far I have used the pair in the same sense, but necessitated by the extension into the case of quaternions in general, I propose now to define ζ-pairs, η-pairs, and ξ-pairs in the forms,

$$\left.\begin{aligned} Q(\zeta, \zeta) &= Q({}_{K\rho}\nabla_1, \rho_1) = Q(K\rho_1, {}_\rho\nabla_1) = \text{etc.}, \\ Q(\eta, \eta) &= Q({}_{Kq}\nabla_1, q_1) = Q(Kq_1, \ \nabla_1) = \text{etc.}, \\ Q(\xi, \xi) &= Q({}_q\nabla_1, q_1) \ \ = Q(q_1, {}_q\nabla_1) , \end{aligned}\right\} \qquad (90)$$

where q and ρ are any quaternion and vector, and McAulay's ζ-pair is the negative of the same pair here. Employing a unit rectangular system, we have

$$\left.\begin{aligned} Q(\zeta, \zeta) &= \qquad\quad\; - Q(i, i) - Q(j, j) - Q(k, k) , \\ Q(\eta, \eta) &= Q(1,1) - Q(i, i) - Q(j, j) - Q(k, k) , \\ Q(\xi, \xi) &= Q(1,1) + Q(i, i) + Q(j, j) + Q(k, k) . \end{aligned}\right\} \qquad (91)$$

For the relation of these three pairs, we have

$$\left.\begin{aligned} Q(\eta, \eta) = Q(1, 1) + Q(\zeta, \zeta) &= Q(\xi, K\xi) , \\ Q(\xi, \xi) = Q(1, 1) - Q(\zeta, \zeta) &= Q(\eta, K\eta) , \\ Q(\eta, \eta) + Q(\xi, \xi) &= 2Q(1, 1) , \\ Q(\eta, \eta) - Q(\xi, \xi) &= 2Q(\zeta, \zeta) . \end{aligned}\right\} \qquad (92)$$

* Utility of Quaternions, etc., Arts. 1–4.

Thus we obtain at once

$$\left.\begin{aligned} S\zeta\zeta &= 3\,, & V\zeta\zeta &= 0\,, \\ S\eta\eta &= 4\,, & V\eta\eta &= 0\,, \\ S\xi\xi &= -2\,, & V\xi\xi &= 0\,. \end{aligned}\right\} \tag{93}$$

The operators $\zeta S\zeta\,(\,)$, $\eta S\eta\,(\,)$ operated on any vector and quaternion respectively produce the same vector and quaternion, while $\xi S\xi\,(\,)$ produces the conjugate; thus,

$$\left.\begin{aligned} \zeta S\zeta\sigma &= \sigma, \\ \eta S\eta p &= p, \\ \xi S\xi p &= Kp. \end{aligned}\right\} \tag{94}$$

The operators $\zeta V\zeta\,(\,)$, $\eta V\eta\,(\,)$, $\xi V\xi\,(\,)$ produce twice the vector, thrice the vector part, and the conjugate of the vector part respectively; thus,

$$\left.\begin{aligned} \zeta V\zeta\sigma &= 2\sigma\,, \\ \eta V\eta p &= 3\,Vp\,, \\ \xi V\xi p &= K\,Vp = -\,Vp\,. \end{aligned}\right\} \tag{95}$$

The operators $S\zeta\rho S\zeta\,(\,)$, $S\eta q S\eta\,(\,)$, $S\xi q S\xi\,(\,)$ produce scalars of the product; thus,

$$\left.\begin{aligned} S\zeta\rho S\zeta\sigma &= S\rho\sigma\,, \\ S\eta q S\eta p &= Sqp\,, \\ S\xi q S\xi p &= SqKp\,. \end{aligned}\right\} \tag{96}$$

The operators $S\zeta\rho\,V\zeta\,(\,)$, etc., produce the scalars of the products of vector parts; thus,

$$\left.\begin{aligned} S\zeta\rho\,V\zeta\sigma &= -\,2S\rho\sigma\,, \\ S\eta q\,V\eta p &= -\,S\,Vq\,Vp\,, \\ S\xi q\,V\xi p &= 3S\,Vq\,Vp\,. \end{aligned}\right\} \tag{97}$$

The operators $V\zeta\rho S\zeta\,(\,)$, etc., produce vectors of the products; thus,

$$\left.\begin{aligned} V\zeta\rho S\zeta\sigma &= V\sigma\rho\,, \\ V\eta q S\eta p &= -\,VKqKp = Vpq\,, \\ V\xi q S\xi p &= -\,VKqp = VKpq\,. \end{aligned}\right\} \tag{98}$$

The operator $V\zeta\rho V\zeta\,(\;)$ produces zero ; thus,

$$V\zeta\rho V\zeta\sigma = 0\,.$$

The operators $\zeta\rho S\zeta\,(\;)$ produce products ; thus,

$$\left.\begin{aligned} \zeta\rho S\zeta\sigma &= \sigma\rho\,, \\ \eta q S\zeta p &= pq\,, \\ \xi q S\xi p &= Kpq\,. \end{aligned}\right\} \tag{99}$$

The operators $S\zeta\,(\;)\,\zeta$ and $V\zeta\,(\;)\,\zeta$, etc., produce upon any quaternion q and ρ, the following results :—

$$\left.\begin{aligned} S\zeta q\zeta &= 3Sq\,, & S\zeta\rho\zeta &= 0\,, \\ S\eta q\eta &= 4Sq\,, & S\eta\rho\eta &= 0\,, \\ S\xi q\xi &= -2Sq\,; & S\xi\rho\xi &= 0\,, \end{aligned}\right\} \tag{100}$$

$$\left.\begin{aligned} V\zeta q\zeta &= -Vq\,, & V\zeta\rho\zeta &= -\rho\,, \\ V\eta q\eta &= 0\,, & V\eta\rho\eta &= 0\,, \\ V\xi q\xi &= 2Vq\,; & V\xi\rho\xi &= 2\rho. \end{aligned}\right\} \tag{101}$$

Cases in which more than one pair of ζ, η, and ξ occur are rather interesting when reduced to simpler cases ; thus, if Q is any function linear in each of the arguments,

$$\left.\begin{aligned} Q(\zeta_1, \zeta_2)\,S\zeta_1\zeta_2 &= Q(\zeta_1, \zeta_2)\,S\zeta_2\zeta_1 = Q(\zeta, \zeta)\,, \\ Q(\eta_1, \eta_2)\,S\eta_1\eta_2 &= Q(\eta_1, \eta_2)\,S\eta_2\eta_1 = Q(\eta, \eta)\,, \\ Q(\xi_1, \xi_2)\,S\xi_1\xi_2 &= Q(\xi_1, \xi_2)\,S\xi_2\xi_1 = Q(\eta, \eta)\,. \end{aligned}\right\} \tag{102}$$

If Q is a linear function in $\zeta_1\zeta_2$, $\eta_1\eta_2$, ξ_1,ξ_2 in combination, then

$$\left.\begin{aligned} Q(\zeta_1\zeta_2)\,V\zeta_1\zeta_2 &= -2Q(\zeta)\,\zeta, \\ Q(\eta_1\eta_2)\,V\eta_1\eta_2 &= 0\,, \\ Q(\xi_1\xi_2)\,V\xi_1\xi_2 &= -4Q(\zeta)\,\zeta; \\ Q(\zeta_1\zeta_2)\,V\zeta_2\zeta_1 &= 2Q(\zeta)\,\zeta, \\ Q(\eta_1\eta_2)\,V\eta_2\eta_1 &= 4Q(\zeta)\,\zeta, \\ Q(\xi_1\xi_2)\,V\xi_2\xi_1 &= 0\,. \end{aligned}\right\} \tag{103}$$

Thus, if Q be linear in the combination $\zeta_1 q \zeta_2$, where q is any quaternion,

$$\left.\begin{aligned} Q(\zeta_1 q \zeta_2)\, S\zeta_1\zeta_2 &= Q(\zeta_1 q \zeta_2)\, S\zeta_2\zeta_1 = 3Q(Sq) - Q(Vq), \\ Q(\eta_1 q \eta_2)\, S\eta_1\eta_2 &= Q(\eta_1 q \eta_2)\, S\eta_2\eta_1 = 4Q(Sq), \\ Q(\xi_1 q \xi_2)\, S\xi_1\xi_2 &= Q(\xi_1 q \xi_2)\, S\xi_2\xi_1 = 4Q(Sq). \end{aligned}\right\} \quad (104)$$

$$\left.\begin{aligned} Q(\zeta_1 q \zeta_2)\, V\zeta_1\zeta_2 &= 2Q(1)\, Vq + 2Q(\zeta)\, \zeta Sq, \\ Q(\eta_1 q \eta_2)\, V\eta_1\eta_2 &= 4Q(1)\, Vq + 4Q(\zeta)\, \zeta Sq. \\ Q(\xi_1 q \xi_2)\, V\xi_1\xi_2 &= 0. \end{aligned}\right\} \quad (105)$$

$$\left.\begin{aligned} Q(\zeta_1 q \zeta_2)\, V\zeta_2\zeta_1 &= -2Q(1)\, Vq - 2Q(\zeta)\, \zeta Sq, \\ Q(\eta_1 q \eta_2)\, V\eta_1\eta_2 &= 0, \\ Q(\xi_1 q \xi_2)\, V\xi_2\xi_1 &= -4Q(1)\, Vq - 4Q(\zeta)\, \zeta Sq. \end{aligned}\right\} \quad (106)$$

The definitions of two pairs of ζ, η, ξ can be at once inferred from those of one pair, and worked thus far on that assumption, but they are written in the forms for any function Q, linear in each of the arguments,

$$\left.\begin{aligned} Q(\zeta_1, \zeta_1, \zeta_2, \zeta_2) &= Q({}_\rho\nabla_1, \rho_1, {}_\sigma\nabla_2, \sigma_2) = Q(\rho_1, {}_\rho\nabla_1, \sigma_2, {}_\sigma\nabla_2) = \text{etc.}, \\ Q(\eta_1, \eta_1, \eta_2, \eta_2) &= Q({}_{Kq}\nabla_1, q_1, {}_{Kp}\nabla_2, p_2) = Q(q_1, {}_{Kq}\nabla_1, p_2, {}_{Kp}\nabla_2) = \text{etc.}, \\ Q(\xi_1, \xi_1, \xi_2, \xi_2) &= Q({}_q\nabla_1, q_1, {}_p\nabla_2, p_2) = Q(q_1, {}_q\nabla_1, p_2, {}_p\nabla_2) = \text{etc.}, \end{aligned}\right\} \quad (107)$$

where ρ and σ are any vectors, which may be the same or different; p and q may also be any quaternions.

Any self-conjugate linear functions may be written

$$\left.\begin{aligned} \bar{\varphi}\rho &= a_1 \zeta S\zeta\rho, \\ \bar{\Phi} q &= a_1 \eta S\eta q, \end{aligned}\right\} \quad (108)$$

where the suffix of the scalar a is supposed to vary with ζ and η.

Any linear function may be written in the form

$$\left.\begin{aligned} \varphi\rho &= \varphi\zeta S\zeta\rho, \\ \Phi q &= \Phi\eta S\eta q. \end{aligned}\right\} \quad (109)^*$$

* Hamilton's Elements, Art. 365.

Their conjugate functions may be written

$$\left.\begin{aligned} \varphi'\rho &= \zeta S\rho\varphi\zeta\,, \\ \Phi' q &= \eta Sq\Phi\eta\,. \end{aligned}\right\} \tag{110}$$

The linear vector function and its conjugate derived from a linear quaternion function may be written

$$\left.\begin{aligned} \varphi\rho &= V\Phi\zeta S\zeta\rho\,, \\ \varphi'\rho &= \zeta S\rho\Phi\zeta\,. \end{aligned}\right\} \tag{111}$$

The invariants and rotation vectors of φ (and of its Hamiltonian inverse function) may be written,

$$\left.\begin{aligned} m &= \tfrac{1}{3}S\varphi\zeta\psi'\zeta\,, \\ m' &= S\zeta\psi\zeta\,, \\ m'' &= S\zeta\varphi\zeta\,; \\ 2\gamma &= -V\zeta\varphi\zeta = V\zeta\varphi'\zeta\,, \\ 2\delta &= -V\zeta\psi\zeta = V\zeta\psi'\zeta\,. \end{aligned}\right\} \tag{112}$$

If φ and π are any two linear vector functions,

$$\left.\begin{aligned} S\varphi\zeta\pi\zeta &= S\pi_\rho\nabla_1\varphi\rho_1 = S\varphi_\rho\nabla_1\pi\rho_1 = S_\rho\nabla_1\varphi'\pi\rho_1 = S_\rho\nabla_1\pi'\varphi\rho_1 \\ &= S\rho_1\varphi'\pi_\rho\nabla_1 = S\rho_1\pi'\varphi_\rho\nabla_1 \\ V\varphi\zeta\pi\zeta &= V\varphi_\rho\nabla_1\pi\rho_1 = -V\pi_\rho\nabla_1\varphi\rho_1 \\ &= V\varphi\rho_1\pi_\rho\nabla_1 = -V\pi\rho_1\varphi_\rho\nabla_1. \end{aligned}\right\} \tag{113}$$

Thus, if

$$Q\,(q,\,q,\,q,\,q,\,\ldots)$$

be any quaternion function linear and homogeneous in q, of the nth degree, then

$$\frac{1}{n!}\,(Sp_{Kq}\nabla_1)^n\,Q_1\,(q,\,q,\,q,\,q,\,\ldots) = Q\,(\,p,\,p,\,p,\,p,\,\ldots)\,, \tag{114}$$

where p is any quaternion constant with respect to q, since

$$\begin{aligned} (Sp_{Kq}\nabla_1)\,Q_1\,(q,\,q,\,q,\,\ldots) &= Q\,(\eta Sp\eta,\,q,\,q,\,\ldots) + Q\,(q,\,\eta Sp\eta,\,q,\,\ldots) \\ &\quad + Q\,(q,\,q,\,\eta Sp\eta,\,q,\,\ldots) + \ldots\,(n \text{ terms}). \end{aligned}$$

When these are operated on again by $Sp_{Kq}\nabla$ each term produces $n - 1$ terms, and so on, until finally after the nth operation, all q's within Q are changed into p's; hence the result. Thus, for the function Q, which is linear in q or ρ, or homogeneous of the nth degree in q or ρ, we obtain, if α and a are respectively a constant vector and a constant quaternion,

$$\left.\begin{aligned} (-1)^n \frac{1}{n!}(S\alpha_\rho\nabla)^n Q(\rho) &= Q(\alpha), \\ \frac{1}{n!}(Sa_{Kq}\nabla)^n Q(q) &= Q(a), \\ \frac{1}{n!}(Sa_q\nabla)^n Q(q) &= Q(Ka). \end{aligned}\right\} \tag{115}$$

This is a generalization of a problem in Tait's Quaternions.*

Hamilton† had proved that if dq is a constant quaternion different from zero such that $d^2q = 0$, $d^3q = 0$, ..., then

$$e^d f(q) = f(q + dq),$$

where f is *any* function, provided the first m differentials do not become infinite. But since the operator $Sp_{Kq}\nabla$, in which p is a constant with respect to q, is equivalent to the operator d with the condition added that $dq =$ the constant p, we may write in our notation the Taylor's Theorems in the forms:

$$\left.\begin{aligned} e^{Sp_{Kq}\nabla} f(q) &= f(q + p), \\ e^{Sp_q\nabla} f(q) &= f(q + Kp), \\ e^{-S\sigma_\rho\nabla} f(\rho) &= f(\rho + \sigma).* \end{aligned}\right\} \tag{116}$$

Similarly

$$\left.\begin{aligned} (e^{S_{Kp}\nabla_{Kq}\nabla} = e^{S_p\nabla_q\nabla}) f(q) F(p) &= f(q + {}_{Kp}\nabla_1) F(p_1) = f(q_1) F(p + {}_{Kq}\nabla_1), \\ (e^{S_p\nabla_{Kq}\nabla} = e^{S_{Kp}\nabla_q\nabla}) f(q) F(p) &= f(q + {}_p\nabla_1) F(p_1) = f(q_1) F(p + {}_q\nabla_1), \\ e^{-S_\sigma\nabla_\rho\nabla} f(\rho) F(\sigma)^* &= f(\rho + {}_\sigma\nabla_1) F(\sigma_1) = f(\rho_1) F(\sigma + {}_\rho\nabla_1), \end{aligned}\right\} \tag{117}$$

where p and q, σ and ρ are independent of each other; and by changing the signs of p, σ, ${}_{Kp}\nabla$, ${}_p\nabla$, ${}_\sigma\nabla$ in both members we have the differences instead of sums in the arguments of the functions f and F.

* 3rd ed., pp. 420, 400, 399.

† Elements, Art. 342.

In the last section we gave the notation

$$d_{\|\alpha} = Sd_{\|\alpha}\rho\ {}_{K\rho}\nabla = Sd\rho\ {}_{K\rho}\nabla_{\|\alpha}\,,$$

$$= S\alpha\ {}_{K\rho}\nabla\ S\alpha^{-1}d\rho = \ldots,$$

to mean that it gives by operation on a function of ρ the differential of the same function when the variable ρ moves parallel to a given vector α; if the vector $d\rho$ is not only parallel to α, but equal in tensor to the same, then the above becomes $S\alpha\,{}_{K\rho}\nabla$, whatever length the vector α may have. But, $S\alpha\,{}_{K\rho}\nabla$ is $Sd\rho\,{}_{K\rho}\nabla = d$, in which the arbitrary vector $d\rho$ is changed to a constant vector α; hence the perfect accordance of both notations. Here it might be remarked also, that if $S\alpha\,{}_{K\rho}\nabla$ is operated only once, then α may also be any function of ρ.

NEW HAVEN, *October* 25, 1895.

THE NON-REGULAR TRANSITIVE SUBSTITUTION GROUPS WHOSE ORDER IS THE CUBE OF ANY PRIME NUMBER.

By Dr. G. A. Miller, Paris, France.

1. The regular groups of order p^3 (p being any prime number) were published almost simultaneously by Young,* Cole and Glover,† and Hölder.‡ At a considerably earlier date Cayley had published these groups for the special case when $p = 2$.‖ It is the object of this paper to determine the remaining transitive groups of this order.

We shall find that there are always two such groups when p is odd. For the special case when p is even it is well known that there is only one.§ Since there are five regular groups of the given order, for every value of p, these facts prove that *the number of the transitive groups of order p^3 is 6 or 7 as p is even or odd.*

The degree, n, ($n < p^3$) of a non-regular transitive group of order p^3 must evidently satisfy the following conditions:

$$p^3 = \alpha n\,, \quad n\,! = \beta p^3. \quad (\alpha, \beta = \text{positive integers})$$

Hence $n = p^2$. Each one of these groups contains a substitution of order p which is commutative to all its substitutions.¶ Since the cycles of this substitution may be regarded as systems of non-primitivity we see that all these groups are non-primitive.**

The degree of the transitive group which corresponds to the permutations of the systems of non-primitivity when one of these groups is transformed by itself is p. Since the order of this group must be a sub-multiple of p^3 as well as of $p\,!$ the group must be regular. Hence every non-regular transitive group of order p^3 contains a subgroup of order p^2 which does not interchange the systems of non-primitivity.

The problem of constructing all the possible non-regular transitive groups of order p^3 may therefore be divided into the following two parts: 1) The construction of the intransitive groups of order p^2 which may be used as sub-

* American Journal of Mathematics, XV, p. 133.

† Ibid., p. 196.

‡ Mathematische Annalen, XLIII, p. 371.

‖ Philosophical Magazine, XVIII, p. 34; cf. ibid., VII, pp. 40, 408.

§ Serret, Liouville's Journal, 1850, p. 52.

¶ Sylow, Mathematische Annalen, V, 588.

** The transitive groups of order p^α are always non-primitive when $\alpha > 1$.

groups, and 2) The construction of the transitive groups of order p^3 which contain these subgroups. We shall consider these parts separately.

2. *The intransitive groups of order p^2 which may be used as subgroups.*

We shall represent the systems of intransitivity of such a group (H) by

$$A_1, A_2, A_3, \ldots, A_p.$$

If one of the substitutions (S) of H would not contain the two systems A_α and A_β ($\alpha, \beta \overline{\gtrless} p$) there would be a transform of S with respect to some substitution in the group of order p^3 which would not contain A_α. Since this transform would contain at least one system (A_γ) which is not found in S, it and S would generate a group of order p^2 whose degree would be less than p^2. All the substitutions of H (excluding identity) must therefore contain either all the systems of intransitivity or all with the exception of one.

The average number of elements in the substitutions of H is $p\,(p-1)$.* We have just proved that this is also the smallest number of elements that can occur in any substitution of H with the exception of identity. Hence H contains $p-1$ substitutions of degree p^2, $p\,(p-1)$ of degree $p\,(p-1)$, and identity.

If we use one of these substitutions (S_1) of degree p^2 as the first generating substitution of H, the cycles of the pth order in a second generating substitution must be the p different powers of the corresponding cycles in S_1. Hence there is one and only one H for each value of p. As this H has been completely determined it remains only to consider the construction of

3. *The transitive groups of order p^3 which contain H.*

If we multiply H by a substitution (S_2) which only interchanges its systems according to a cycle of order p we obtain a group (G_1) which contains only substitutions of the pth order and identity.† If we multiply H by sS_2 (s being any cycle of S_1) we obtain another group (G_2) which contains $(p+1)(p-1)$ substitutions of order p and $p^2(p-1)$ of order p^2. It remains to prove that the other groups (G_ϵ; $\epsilon = 3, 4, \ldots$) are not distinct from these two.

Every G is commutative to S_1. We therefore obtain each one of the possible G's once and only once if we multiply H by

$$s_1^{a_1} s_2^{a_2} s_3^{a_3} \ldots s_{p'}^{a_{p'}} S_2, \quad (a_1, a_2, a_3, \ldots, a_{p'} = 1, 2, 3, \ldots, p)$$

where $s_1, s_2, s_3, \ldots, s_{p'}$ are the cycles of S_1, in order, and $p' = p - 2$.

Since

$$(s_1^{x_1} s_2^{x_2} \ldots s_p^{x_p})^{-1} S_2 s_1^{x_1} s_2^{x_2} \ldots s_p^{x_p} = s_1^{x_2 - x_1} s_2^{x_3 - x_2} \ldots s_p^{x_1 - x_p} S_2,$$

* Frobenius, Crelle's Journal, CI, p. 287.

† The only exception occurs when $p = 2$. For this value of p G_1 and G_2 are the same group.

and the system of indeterminate equations

$$x_2 - x_1 = a_1, \quad x_3 - x_2 = a_2, \quad \ldots, \quad x_1 - x_p = a_p$$

can always be solved when

$$a_1 + a_2 + a_3 + \ldots + a_p = 0,$$

all of these groups must be conjugate to

$$Hs_1^{a_1}S_2 \qquad\qquad (a_1 = 1, 2, \ldots, p)$$

with respect to substitutions of the form

$$s_1^{a_1}s_2^{a_2} \ldots s_p^{a_p}.$$

When $a_1 = 1$, $Hs_1^{a_1}S_2 = G_2$; when $a_1 = p$, $Hs_1^{a_1}S_2 = G_1$; for the other values of a_1, $Hs_1^{a_1}S_2$ is evidently conjugate to G_2 with respect to the substitutions which transform S_1 into its different powers without interchanging its cycles. Hence G_1 and G_2 are the only non-regular groups of order p^3.

April, 1896.

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